

# Renormalization group and bound states \*

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Renormalization group procedure suggests that the low-energy behavior of effective coupling constant in asymptotically free Hamiltonians is connected with the existence of bound states and depends on how the interactions responsible for the binding are included in the renormalization group equations.

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## I. INTRODUCTION

Using a very simple model with asymptotic freedom (or limit cycle) and bound states, this lecture explains a renormalization group (RG) procedure for Hamiltonians, including the result that an effective interaction strength grows at low energies. Instead of integrating out high-energy degrees of freedom, the procedure explained here is carried out using similarity transformations. The magnitude of increase of the interaction strength (a coupling constant in an effective Hamiltonian) at low energies is related in the model to the existence of a bound state and a degree to which the interaction responsible for the effect of binding is included in the generator of the similarity transformation. Most briefly, the more interaction in the generator the less increase in the coupling constant. In addition, since a tiny and quite generic modification of the simple model changes the asymptotically free behavior into a limit cycle of a large period, the model shows that an apparently asymptotically free behavior in a considered range of scales may actually be a part of a cycle with a period much larger than the considered range. In the limit cycle case, the model illustrates how a large set of bound states can influence behavior of the effective coupling constant at low energies, also depending on the generator used for the similarity transformation. Key references for this lecture are [1] and [2].

How the renormalization group findings concerning bound states may be related to the AdS/CFT correspondence [3, 4], including the RG interpretation proposed by Polyakov [5, 6], is not yet established. On the other hand, the RG procedure discussed here is particularly useful in the front form of Hamiltonian dynamics [7, 8]. This form of dynamics is precisely the one in which recent holographic models have been proposed for calculating masses and wave functions of bound states of quarks and gluons [9, 10].

Section II defines the model Hamiltonian. Subsequent sections introduce relevant concepts on the basis of elementary calculations in the model. Ultraviolet divergences are identified in Section III. Section IV reviews a standard RG procedure. The review includes: derivation of the result that a coupling constant increases at low energies (without explicit discussion of scattering processes or Green's functions); demonstration of asymptotically free behavior of the coupling constant; explanation of how counterterms are calculated; and identification of difficulties associated with the increase of the coupling constant at low energies. Section V introduces the similarity RG (SRG) procedure. A simple class of generators of similarity transformations is introduced in Section V A. Solutions for effective Hamiltonians in the SRG procedure with different generators, are described in Section VI. The case of asymptotic freedom is discussed in Section VI A, and the case of limit cycle, related to asymptotic freedom, is discussed in Section VI B. Section VII concludes the lecture by a summary of the connection found in the model between the increase of effective interaction strength at small energies and existence of bound states.

A set of brief appendices is added in order to point out analogies between the simple model and quantum field theory (QFT), such as QCD. The appendices discuss: singular,  $\delta$ -function potentials in effective Schrödinger equations (Section VIII A), theory of effective particles that aims at explaining the origin of the constituent quark model and binding of partons (Section VIII B), and some questions concerning a possible connection between the SRG procedure and AdS/CFT correspondence (Section VIII C).

## II. MODEL

In order to understand the mechanism that causes an asymptotically free interaction to rapidly increase at small energies, a theoretical feature often suggested to be associated with confinement, we need a precise definition of what is meant by the effective coupling constant. In theories as complex as QCD, the concept of effective coupling constant

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at low energies is not simple. Therefore, it seems appropriate to first define the concept using a simple model and see what happens there before one plunges into the complexity of QCD.

The well-known asymptotically free coupling constant depends on the energy scale logarithmically (it is proportional to the inverse of the logarithm of the energy). The energy is actually a kinetic one, not involving interactions in any significant way, and the scale should be conceptually associated with momentum, a kinematical variable of quantum mechanics rather than a dynamical one. Thus, when constructing a simple model Hamiltonian in the generic form of

$$H = H_0 + H_I, \quad (1)$$

one considers  $H_0$  a free (kinetic) energy and  $H_I$  an interaction. An alternative way of thinking is that all eigenvalues and eigenstates of  $H_0$  are known and provide a basis for describing what happens due to  $H_I$ . In both ways, for studies of logarithmic effects of asymptotic freedom, it is useful to assume that the spectrum of  $H_0$  is not degenerate and has the form

$$H_0|n\rangle = E_n|n\rangle, \quad E_n = \mu b^n, \quad b > 1, \quad (2)$$

where  $\mu$  is a unit of energy. For convenience,  $\mu$  is set to 1 and omitted from further consideration (all numbers that refer to energy need to be multiplied by  $\mu$  in order to obtain the actual quantity). The utility of using powers of  $b$  is that successive momentum scales, or eigenvalues of  $H_0$ , are separated by a constant on a logarithmic scale, the constant being  $\ln b$ . Therefore, a small number of states, just one per momentum (or energy measured by  $H_0$ ) scale, will be sufficient to track logarithmic effects associated with asymptotic freedom and, at the same time, control effects of binding. Once the eigenstates of  $H_0$  are normalized,  $\langle m|n\rangle = \delta_{mn}$ , the matrix elements of  $H_0$  in the model are

$$\langle m|H_0|n\rangle = E_m\delta_{mn}. \quad (3)$$

The interaction Hamiltonian  $H_I$  is defined in the model by its matrix elements, using assumption that these elements should be factorized, i.e.,  $H_{Imn} = H_m H_n$ , for the purpose of having very simple exact solutions to the eigenvalue problem for  $H$ . Thus, the factor  $H_n$  should have dimension of square root of energy. The simplest possibility is  $H_n \sim \sqrt{E_n}$ . The proportionality is reduced to a dimensionless number, and one can write

$$\langle m|H_I|n\rangle = -g\sqrt{E_m}\sqrt{E_n}, \quad (4)$$

where  $g$  determines the strength of the interaction. It is called coupling constant, in analogy with the standard nomenclature in QFT. The negative sign is chosen so that there exists a bound state for a sufficiently large positive  $g$ . By definition, the bound state corresponds to a negative eigenvalue of  $H$ .

In summary, matrix elements of the model Hamiltonian are

$$H_{mn} = \langle m|H|n\rangle = E_m\delta_{mn} - g\sqrt{E_m}\sqrt{E_n}. \quad (5)$$

Note that it will have to be clarified what the words “sufficiently large  $g$ ” are supposed to mean, because the model Hamiltonian produces divergences (infinities) no matter how small the number  $g$  is.

### III. ULTRAVIOLET DIVERGENCES

Consider first very small numbers  $g$  and the intuition that eigenvalues of  $H$  should be nearly equal to the eigenvalues of  $H_0$ . For example, first-order perturbation theory produces a correction  $\Delta E_m^{(1)}$  to the energy  $E_m$  of the form

$$\Delta E_m^{(1)} = \langle m|H_I|m\rangle = -g E_m. \quad (6)$$

When  $g$  is small, the correction is small; it is just a fraction  $g$  of the energy being corrected. (This situation resembles what happens in the Schrödinger equation for atoms or positronium when one derives corrections to the Coulomb potential from QED and calculates their influence on energy levels in first-order perturbation theory, see Appendix, Section VIII A.)

Consider now the second-order correction to the same energy

$$\Delta E_m^{(2)} = \sum_{k \neq m} \frac{|\langle m|H_I|k\rangle|^2}{E_m - E_k} = g^2 E_m \sum_{k \neq m} \frac{E_k}{E_m - E_k}. \quad (7)$$

Since  $E_k = b^k$ , a term number  $k$  contributes  $1/(b^{m-k} - 1)$  in the sum. Thus, terms with  $k < m$  contribute a finite sum of a nearly geometric series (with quotient  $1/b$ ), which is not sensitive to any lower bound on  $k$  in the sum, say

$M$ , if  $b^M \ll b^m$ . However, a lower bound is needed to define a Hamiltonian whose matrix has a finite size. In order to focus attention on  $E_m \sim 1$ , one can set the lower bound on  $k$  to be a large negative integer  $M$ . As a result, all eigenvalues of  $H_0$  included in the model are from now on assumed to satisfy the condition  $E_k \geq b^M$ , and  $b^M \ll 1$ .

In contrast, terms with  $k \gg m$  contribute each  $-1$ . For  $b \gg 1$ , the second-order correction is proportional to the number of basis states with energies (eigenvalues of  $H_0$ ) greater than  $E_m$ . In order to obtain a finite answer, one must limit this number. For example, one can limit from above the range of summation over  $k$  by certain large positive integer  $N$ . Then, one has to understand what happens when  $N$  is very large. Imposing a limit such as  $N$  is called ultraviolet regularization.

Including the ultraviolet regularization, the result of Eq. (7) is approximated by the formula (the larger  $b$ , the better the approximation)

$$\Delta E_m^{(2)} = -g^2 E_m (N - m). \quad (8)$$

This formula shows that the correction tends to  $-\infty$  when one attempts to send the ultraviolet cutoff on energies,  $\Lambda = b^N$ , to infinity. Since  $N = \ln \Lambda / \ln b$ , the second-order correction is also approximately given by the formula

$$\Delta E_m^{(2)} = -\frac{g^2 E_m}{\ln b} \ln \frac{\Lambda}{E_m}. \quad (9)$$

This result explains why the correction is called logarithmically divergent in the ultraviolet. A direct comparison of Eqs. (8) and (9) illustrates that the ultraviolet logarithmic divergence results from all different energy scales contributing equally to the eigenvalues. This conclusion is not limited to perturbation theory.

Consider the eigenvalue problem for the matrix  $[H_{mn}]$ ,

$$\sum_{n=M}^N H_{mn} \psi_n = E \psi_m, \quad (10)$$

whose subscripts are limited after the regularization,  $M \leq m, n \leq N$ . Because the interaction is factorized, one has

$$\psi_m = \frac{g\sqrt{E_m}}{E_m - E} \sum_{n=M}^N \sqrt{E_n} \psi_n \quad (11)$$

and the sum in Eq. (11) is just a number (does not depend on  $m$ ), say  $c$ . Substituting this solution for the wave function into Eq. (10), one obtains

$$E_m \frac{cg\sqrt{E_m}}{E_m - E} - g\sqrt{E_m} \sum_{n=M}^N \frac{cgE_n}{E_n - E} = E \frac{cg\sqrt{E_m}}{E_m - E}. \quad (12)$$

or

$$1 + g \sum_{n=M}^N \frac{E_n}{E - E_n} = 0. \quad (13)$$

As far as the high-energy part is concerned, the sum in this non-perturbative eigenvalue condition is of the same type as in the perturbative Eq. (7), the only change being that  $E_m$  with one particular value of  $m$  is now replaced by the unknown eigenvalue  $E$ . This does not change the fact that the sum can be approximated by  $-\ln(\Lambda/|E|)$ . The result is that the eigenvalue condition has no meaning for finite  $E$  when  $\Lambda \rightarrow \infty$ .

More specifically, consider a possibility that a bound state exists, with a negative eigenvalue  $E = -E_B$ . Replace the sum over  $n$  by an integral with measure  $dn = dE_n/(E_n \ln b)$ , just to get an idea what happens. One has

$$1 - \frac{g}{\ln b} \int_{b^M}^{\Lambda} \frac{dE}{E_B + E} = 0, \quad (14)$$

or

$$E_B = \frac{\Lambda - b^M e^{\frac{\ln b}{g}}}{e^{\frac{\ln b}{g}} - 1}. \quad (15)$$

When one takes the limit of  $\Lambda \rightarrow \infty$  for fixed  $g$ , the binding energy diverges linearly with  $\Lambda$ .

One can also observe that the square of the matrix  $H_I$  with large  $N$  and  $M$  is equal to  $\frac{gb\Lambda}{1-b}H_I$ , which means that it diverges in the ultraviolet limit of  $\Lambda \rightarrow \infty$  for fixed  $g$ . This means that all powers of the entire  $H$  are ultraviolet divergent. In particular, the evolution operator  $U(t, 0) = e^{-iHt}$  does not exist in this limit.

Note that the divergences in Eqs. (8), (9), and (15), result from the diverging number of degrees of freedom. Similar problems may occur in classical statistical systems with a huge number of degrees of freedom if all these degrees of freedom contribute significantly to observables [11].

#### IV. STANDARD RENORMALIZATION GROUP PROCEDURE

Given the divergences, the task now is to figure out whether to keep or discard the model that produces such diverging results when  $N$  grows. Of course, one can keep  $N$  fixed and try to describe physics (in the simple model, “physics” amounts to a set of eigenvalues and transition amplitudes). But if  $N$  is expected (and hoped) to be very large (a desire a physicist has when designing a fundamental theory), the divergences can only be removed by making the coupling constant  $g$  vanishingly small, and fine tuned to the value of  $\Lambda = b^N$ . On the other hand, a good model is expected to capture physics in a natural way, without worrying about very large numbers that by no means are comparable to the scale of observables of immediate interest. So, the trouble with the diverging model can be summarized as follows.

Initially (in the model, by assumption; in realistic theories, motivated by experience gathered through observations and experiments), one is led to believe that certain  $H_I$  is a good candidate to consider. For example, in the model one may imagine that one measures transition rates between two states, say  $|m_1\rangle$  and  $|m_2\rangle$ , and that these rates can be described in first-order perturbation theory by an interaction Hamiltonian with matrix elements

$$\begin{bmatrix} \langle m_2 | H_I | m_2 \rangle, & \langle m_2 | H_I | m_1 \rangle \\ \langle m_1 | H_I | m_2 \rangle, & \langle m_1 | H_I | m_1 \rangle \end{bmatrix} = -g \begin{bmatrix} E_{m_2}, & \sqrt{E_{m_2} E_{m_1}} \\ \sqrt{E_{m_1} E_{m_2}}, & E_{m_1} \end{bmatrix}. \quad (16)$$

One is then compelled to postulate that the whole matrix of  $H_I$  has the form given in Eq. (4). This way of thinking leads to divergences, as described in the previous section, and the question what to do about them. It is clear now that one does not want to abandon the proposed interaction entirely since it does work for small  $g$  in the cases of interest (in the example, for evolution of states built from  $|m_1\rangle$  and  $|m_2\rangle$ ) and exhibits appealing symmetry. In fact, this way of thinking is used in building theories, by extrapolation from known examples. An analogy in QFT is provided by the gauge symmetry [12]. Thus, one needs to conceive a general way out of the problem with divergences that are produced by naive extrapolation of knowledge from a small set of matrix elements to a large set of them. The large set is desired when one seeks a theory of presumably large range of applicability and a lot of predictive power.

The basic idea described here (using the model) has been originally formulated in Refs. [13, 14]. The idea is to learn what happens when one starts with some large  $N$  and tries to reduce the value of  $N$  to a smaller value, say  $N_1$ , and properly includes all effects due to states between  $N$  and  $N_1$ . (This is sometimes called “integrating out high-energy degrees of freedom.”) The resulting Hamiltonian  $H^{(1)}$ , limited in energy by  $\Lambda_1 = b^{N_1}$ , will contain information about what has to be done in order to compensate for the presence of the arbitrarily chosen cutoff  $\Lambda_1$ . The step of reducing the size of a cutoff can be repeated. One can reduce  $\Lambda_1$  to  $\Lambda_2$ ,  $\Lambda_2$  to  $\Lambda_3$ , and so on. Such steps are called RG transformations. When a transformation is applied  $K$  times, a chain of Hamiltonians is obtained, including  $H$ ,  $H^{(1)}$ , ...,  $H^{(K)}$ . Eventually, two things happen.

First, the relationship between  $H^{(K)}$  and  $H^{(K+1)}$  may become universal in the sense that it no longer depends on all details of the initial  $H$ . For example, if  $H^{(K)}$  contains an interaction term of the form  $-g^{(K)}\sqrt{E_m E_n}$  and  $H^{(K+1)}$  contains an interaction term of the form  $-g^{(K+1)}\sqrt{E_m E_n}$ , the relationship between  $g^{(K)}$  and  $g^{(K+1)}$  may be independent of the value of the initial coupling constant  $g$  in  $H$ . Instead, a universal recursion is found for the coupling constant when the cutoff is changed. (Such recursion may include rescaling variables in order to compare successive Hamiltonians in terms of functions of dimensionless variables in a fixed range, leading to anomalous dimensions.) We will see how the recursion emerges in the model shortly. The model illustrates this way how the existence of a  $\beta$ -function in QED [15] could be understood, and explained in Ref. [16] in the context of strong interactions. This is also how universality in critical phenomena could be explained in classical statistical mechanics [11].

Second, after many RG steps, one obtains Hamiltonians with running cutoffs  $\Lambda_K$ . Suppose that one reduces the cutoff in every RG step by a factor of  $b$ . Then  $\Lambda_K = b^{N-K}$  may be finite even when  $\Lambda$  is sent to infinity. What is required is that the number of the RG steps,  $K$ , increases when one increases  $N$  for fixed  $N - K$ . Since the finite cutoff  $\Lambda_K$  can be chosen arbitrarily and, by construction, eigenvalues smaller than  $\Lambda_K$  for all  $K$  considered do not depend on  $\Lambda_K$  at all, one eventually obtains a family of effective Hamiltonians,  $H_\lambda$  labeled by a finite cutoff parameter  $\lambda = \Lambda_K$ . Predictions that follow from  $H_\lambda$  do not depend on  $\lambda$ .

### A. Gaussian step

It is time now to attempt the RG procedure described in the previous section in the case of our model. We will see how lowering  $\lambda$  leads to increase of a coupling constant and why this increase causes trouble.

There are essentially two interrelated tasks to accomplish. One is to establish the Hamiltonian with the large cutoff  $N$ , so that the resulting theory does not produce divergences (dependence of observables on  $N$ ). The other one is to evaluate  $H_\lambda$  with a finite  $\lambda$  from the well-defined initial  $H$  with a priori arbitrarily large  $N$ . The conceptual difficulty of the RG procedure is that the first task is accomplished in the process of trying to complete the second one, and in the second task, the result for  $H_\lambda$  is used to decide how the initial  $H$  should be defined in order to make sure that matrix elements of  $H_\lambda$  do not depend on  $\Lambda$ . The need for executing this process in a sequence of successive approximations that can converge on a structure that one is looking for, is the hardest aspect of the procedure to understand.

In the simple model, one can start with the eigenvalue problem

$$H|\psi\rangle = E|\psi\rangle, \quad (17)$$

$$|\psi\rangle = \sum_{k=M}^N \psi_k |k\rangle, \quad (18)$$

in which the coefficients  $\psi_k$  (a wave function) satisfy the set of linear equations (10). One RG step is done by eliminating  $\psi_N$  from the remaining  $N - M$  equations. Note that the initial number of equations is  $N - M + 1$ , because there is one equation with  $E_0 = 1$ , in addition to  $N$  equations corresponding to positive and  $-M$  equations corresponding to negative powers of  $b$ .

The eigenvalue problem is split into one equation for the highest energy component and a set of equations for the remaining components:

$$E_N \psi_N + \sum_{n=M}^N H_{INn} \psi_n = E \psi_N, \quad (19)$$

$$E_m \psi_m + \sum_{n=M}^N H_{Imn} \psi_n = E \psi_m, \quad M \leq m \leq N - 1. \quad (20)$$

Eq. (19) produces

$$\psi_N = (E - E_N - H_{INN})^{-1} \sum_{n=M}^{N-1} H_{INn} \psi_n, \quad (21)$$

which is used in the remaining  $N - M$  equations to eliminate  $\psi_N$  from them. In its essence, the RG step is Gaussian elimination of one equation in a set of linear equations. The set involves the unknown eigenvalue  $E$ . The result of the first step is a set of equations with  $M \leq m \leq N - 1$ ,

$$E_m \psi_m + \sum_{n=M}^{N-1} H_{Imn} \psi_n + \sum_{n=M}^{N-1} \frac{H_{ImN} H_{INn}}{E - E_N - H_{INN}} \psi_n = E \psi_m. \quad (22)$$

Therefore, the new interaction “Hamiltonian” in this one-step smaller set of equations has the following matrix elements ( $M \leq m, n \leq N - 1$ ):

$$H_{Imn}^{(1)} = H_{Imn} + \frac{H_{ImN} H_{INn}}{E - E_N - H_{INN}}. \quad (23)$$

The word “Hamiltonian” is used in quotation marks because the matrix elements of  $H_I^{(1)}$  depend on the unknown eigenvalue  $E$ . Nevertheless, Eq. (23) is an exact result in the model. It guarantees that  $H^{(1)} = H_0 + H_I^{(1)}$  with the cutoff  $\Lambda_1 = b^{N-1}$  has the same eigenvalue  $E$  as the Hamiltonian  $H$  with the cutoff  $\Lambda = b^N$  has. On the other hand, the operator  $H_I^{(1)}$  is not fully defined before one specifies how to find the eigenvalue it depends on. Some consistency conditions would have to be imposed in a way that still allows us finding the initial Hamiltonian and reliably calculate  $H_\lambda$ .

The situation simplifies considerably if one can limit the RG procedure to eigenvalues  $E$  that are much smaller than a suitable finite cutoff  $\lambda$  one wishes to reach. “Suitable finite cutoffs  $\lambda$ ” in realistic theories are cutoffs that are small

enough so that one can solve the eigenvalue problem for  $H_\lambda$  on a computer. This condition puts severe constraints on the size of  $\lambda$  in realistic theories, illustrated by, e.g., computational limitations of the lattice gauge theory. One has to execute  $N - n$  relatively complex RG steps in order to reduce the cutoff from a formally infinite  $\Lambda = b^N$  to some finite  $\lambda$  and obtain  $H_I^{(N-n)}$  in  $H_\lambda$  with  $\lambda = b^n$  that is sufficiently small for a reliable computation of the eigenvalues.  $H_\lambda$  still includes a large number of matrix elements, on the order of  $(n - M + 1)^2$  times a potentially large number of other degrees of freedom besides the size of momentum, such as the numbers of virtual particles and their angular momenta, spins, colors, or flavors.

In order to see how the simplification mentioned above emerges for small eigenvalues  $E$ , and how the simplification eventually ceases to be valid in the case of asymptotically free Hamiltonians, one may step back to Eq. (23) and check what happens in our model. Conclusions will not be limited to the model case.

### B. Asymptotic freedom in the model Hamiltonian

In the model, Eq. (23) reads

$$H_{Imn}^{(1)} = -g\sqrt{E_m E_n} + \frac{(-g\sqrt{E_m E_N})(-g\sqrt{E_N E_n})}{E - E_N + gE_N} \quad (24)$$

$$= -\left(g - \frac{g^2 E_N}{E - E_N + gE_N}\right) \sqrt{E_m E_n}. \quad (25)$$

It is clear that the interaction “Hamiltonian”  $H_I^{(1)}$  has the same structure of matrix elements as  $H_I$  but contains a new “coupling constant,” say  $g^{(1)}$ , that depends on the eigenvalue  $E$ .  $g^{(1)}$  is given by the expression in the bracket in Eq. (25).

The simplification for cutoffs much larger than  $E$  becomes obvious when one re-writes Eq. (25) as

$$g^{(1)} = g - \frac{g^2 E_N}{E - E_N + gE_N} = g \frac{1 - E/E_N}{1 - g - E/E_N}. \quad (26)$$

One can neglect the ratio  $E/E_N$  provided that the eigenvalue  $E$  is small in comparison to the energy  $E_N = b^N$ . With this simplification, Eq. (26) reads

$$g^{(1)} = \frac{g}{1 - g}, \quad (27)$$

and implies the following recursion in further RG steps

$$g^{(K)} = \frac{g^{(K-1)}}{1 - g^{(K-1)}}, \quad (28)$$

for as long as  $E_{N-K} \gg |E|$  and  $1 - g \gg |E|/E_{N-K}$ . The recursion of Eq. (28) is solved by

$$g^{(K)} = \frac{g}{1 - gK}. \quad (29)$$

Suppose one can solve the eigenvalue problem for  $H^{(K)} = H_{\lambda_0}$  with  $\lambda_0 = b^{N-K}$  and establish that the coupling constant  $g(\lambda_0) = g^{(K)}$  should have some value  $g_0$  in order to reproduce some measured eigenvalue  $E_0 \ll \lambda_0$ . Eq. (29) says that

$$g_0 = \frac{g_\Lambda}{1 - \frac{g_\Lambda}{\ln b} \ln \Lambda / \lambda_0}. \quad (30)$$

Eq. (30) allows one to calculate  $g_\Lambda$  that needs to stand in the initial  $H$  with the cutoff  $\Lambda$  to produce the same eigenvalue. The result is

$$g_\Lambda = \frac{g_0}{1 + \frac{g_0}{\ln b} \ln \Lambda / \lambda_0}, \quad (31)$$

This means that the model is asymptotically free: the larger the cutoff  $\Lambda$  in the initial  $H$ , the smaller the coupling constant  $g_\Lambda$  in it. The most famous analogy in QFT is found in Refs. [17, 18].

### C. Calculation of counterterms

Since Eq. (31) has been obtained from an exact RG procedure under the two conditions that  $|E_0| \ll \lambda_0$  and  $1 - g_0 \gg |E_0|/\lambda_0$ , one may expect that the replacement of  $g$  by  $g_\Lambda$  given by Eq. (31) provides a model  $H$  with all eigenvalues  $E$  that satisfy the same two conditions being also independent of  $\Lambda$ . Thus, these eigenvalues are also expected to be common for all Hamiltonians  $H_\lambda$  with  $\lambda \geq \lambda_0$ , i.e., they do not depend on the finite running cutoff  $\lambda$  that limits the momentum range in  $H_\lambda$ .

The model calculation presented so far provides an example of how one takes the first step in identifying the initial  $H$ . The point is that the matrix elements of the initial  $H$  can be now written as

$$H_{mn} = E_m \delta_{mn} - g \sqrt{E_m} \sqrt{E_n} + CT_{mn}, \quad (32)$$

$$CT_{mn} = (g - g_\Lambda) \sqrt{E_m} \sqrt{E_n}, \quad (33)$$

where the letters  $CT$  are chosen as an abbreviation for the word “counterterm.” The counterterm guarantees that small eigenvalues do not depend on the ultraviolet cutoff  $\Lambda$ . The size of the coupling constant  $g$  that was considered small in Section III, can now be understood as actually concerning  $g_\Lambda$  in the initial, regulated Hamiltonian that includes counterterms; a bound state exists if  $g_\Lambda$  is sufficiently large and this means different sizes of  $g_\Lambda$  for different values of  $\Lambda$ .

Note that the finite part of the counterterm, which amounts to the choice of  $g_\lambda$  at some value of  $\lambda$ , is fixed by the condition of matching one eigenvalue with experiment. This one condition by itself (fixing one parameter to match one energy) would not guarantee that other eigenvalues are cutoff independent. But it does provide such guarantee when all matrix elements of the  $CT$  (many numbers) are calculated in the RG procedure and fixing just one value of the coupling constant properly determines the entire Hamiltonian matrix so that the low energy eigenvalues do not depend on the cutoff  $\Lambda$  (and, by construction, also do not depend on the cutoff  $\lambda$ ). One may expect that other matrix structures than just  $\sqrt{E_m E_n}$  emerge in  $H_\lambda$  with small  $\lambda$ , but these structures must cease to depend on  $\Lambda$  when  $\Lambda \rightarrow \infty$  once  $g_\Lambda$  is set according to Eq. (31).

In more detail, the argument that other eigenvalues of  $H_{\lambda_0}$  that are much smaller than  $\lambda_0$  will also be independent of the cutoff  $\Lambda$  once one inserts in the initial  $H$  the counterterm defined in Eq. (33), is following. Eq. (28) is valid and the same for eigenvalues that satisfy the conditions  $|E| \ll \lambda_0$  and  $1 - g_0 \gg |E|/\lambda_0$ . In fact, the exact way in which all matrix elements of  $H^{(K=N-n)}(E)$  in the simple model depend on the initial cutoff  $\Lambda$  is determined by the dependence of  $g^{(K)}$  on the sequence of ratios  $E/\Lambda$ ,  $bE/\Lambda$ , ...,  $b^{N-n}E/\Lambda = E/\lambda$  and the initial value of the coupling constant, i.e., after inclusion of the counterterm, just  $g_\Lambda$ . For as long as these ratios are negligible, all matrix elements of “Hamiltonians”  $H^{(K)}(E)$  do not actually depend on the value of  $E$  and do not depend on the cutoff  $\Lambda$ . Therefore, their eigenvalues are also independent of  $\Lambda$ . However, one cannot claim that all their eigenvalues are the same because the number of eigenvalues depends on the size of the cutoff (the smaller the cutoff the smaller the number of eigenvalues). Only the eigenvalues much smaller than the smallest cutoff  $\lambda$  in the recursion are the same and independent of  $\Lambda$  when calculated from all Hamiltonians  $H_\lambda$  in which the coupling constant changes from one value of  $\lambda$  to another according to the asymptotic freedom formula.

Additional steps in evaluation of counterterms would be required in order to obtain corrections to the leading counterterm in Eq. (33) due to the ratios  $E/\Lambda_K$  differing from 0 in the RG recursion. Readers interested in such steps may consult Ref. [19]. It is not necessary to discuss these additional steps here in order to see the problem with increasing of  $g_\lambda$  when  $\lambda$  decreases.

However, before proceeding to the issue of large coupling constant at low energies (in the next section), it should be mentioned for readers used to thinking about RG in terms of differential equations, that the reduction of one running cutoff on momenta to another one that is smaller, can be made in infinitesimally small steps (in the case of continuous momentum variables). The corresponding changes in the Hamiltonian matrix elements are then described by differential equations.

### D. Difficulty with $g_\lambda$ that grows at small $\lambda$

Consider Eq. (26) which implies the recursion

$$g_{\lambda/b}(E) = g_\lambda(E) \frac{1 - E/\lambda}{1 - g_\lambda(E) - E/\lambda}. \quad (34)$$

As discussed in the previous section, this recursion simplifies enormously when one can disregard the ratio  $E/\lambda$ . And when one does, one obtains the conclusion that  $g_\lambda$  increases when  $\lambda$  decreases (this is equivalent to asymptotic freedom which says that  $g_\lambda$  decreases when  $\lambda$  increases).

The increase of  $g_\lambda$  toward small  $\lambda$  causes a major difficulty because the ratio  $E/\lambda$  is compared with  $1 - g_\lambda$  in the denominator in Eq. (34). When the initial value of  $g_\lambda$  at some large  $\lambda$  is small and  $g_\lambda$  increases when  $\lambda$  decreases, at some point the effective coupling constant approaches 1. At this point, the ratio  $E/\lambda$  cannot be neglected no matter how small it is. In other words, the entire procedure can no longer be based on simplifications that rely on smallness of the eigenvalue in comparison to cutoff.

Not only the simplifications described in the previous section no longer apply, but also the entire procedure becomes unstable because the small ratio occurs in the denominator. The small denominator is reminiscent of small energy denominators that occur in perturbation theory and cause infrared singularities. However, the simple model shows that the problem is related to the strength of interactions that are capable of canceling kinetic energy terms and producing negative elements on the diagonal of the Hamiltonian matrix. Such situation can occur when bound states are formed: attractive (negative) interactions are larger than the kinetic energies. In order to control what happens when  $g_\lambda$  approaches 1, a different RG procedure seems required.

## V. SIMILARITY RENORMALIZATION GROUP PROCEDURE

In the standard RG procedure, one evaluates effective Hamiltonians  $H_\lambda$ , with small, running cutoffs  $\lambda$ , as is illustrated in the simple model in the previous sections. One finds counterterms in the initial Hamiltonian with cutoff  $\Lambda$  by demanding that all matrix elements of the effective  $H_\lambda$  are independent of  $\Lambda$  and by fixing their finite parts by comparison with experiment (the comparison may include conditions of symmetry [20]). Then the effective Hamiltonians can in principle be calculated for sufficiently small  $\lambda$  to carry out computations of their eigenvalues (or other observables) using computers. Unfortunately, for feasibly small cutoffs, the effective coupling constant in asymptotically free theories may be so large that naive expectations based on asymptotic freedom formulae may be inadequate. An alternative approach is offered by the similarity renormalization group (SRG) procedure [21].

In the SRG procedure, one proceeds according to similar principles as in the standard approach described in the previous sections. One also completes two interrelated tasks. In the one task, one finds counterterms. In the other task, one evaluates effective Hamiltonians. The new idea is that one does not “integrate out” any degrees of freedom. Instead, one changes the basis states by rotating them in the Hilbert space. The rotation is designed in such a way that it guarantees the new Hamiltonian  $H_\lambda$  to have vanishing matrix elements between basis states if they differ in energy by more than  $\lambda$ . This means that the matrix elements of  $H_\lambda$  that result from the second task in the SRG procedure, are different from 0 only within an energy band of width  $\lambda$  along the diagonal. The algebra of the procedure is designed in such a way that one never encounters small energy denominators (the differential version of SRG procedure has the same property). Moreover, the SRG procedure allows for direct evaluation of matrix elements of  $H_\lambda$  without knowing anything about eigenvalues. The reader will easily find all required details in the original literature.

The SRG task of evaluating  $H_\lambda$  with small  $\lambda$  aims at evaluating a near-diagonal Hamiltonian matrix in such a way that the calculation may be carried out in perturbation theory with a small error that can decrease when the order of the perturbative calculation increases (perturbative approach is required in realistic theories due to their complexity that initially cannot be handled in any other way). One may not apply perturbation theory to complete diagonalization, because this would involve solving also for non-perturbative features such as bound-states. However, the SRG procedure can produce a Hamiltonian matrix of a small width  $\lambda$  by “rotating out” only those interactions that involve energy changes larger than  $\lambda$  and can be treated in perturbation theory. The resulting  $H_\lambda$  must be diagonalized on a computer. The smaller  $\lambda$ , the smaller the space of states required to find the spectrum in the range of interest. But the smaller  $\lambda$ , the higher order of perturbative SRG evolution is required for accuracy. A compromise must be found, and this is not easy in QCD (see Appendix, Section VIII B). But there is also a hope that already first 4 orders of the calculation will be sufficient to identify key operator structures in  $H_\lambda$ .

Of course, the SRG procedure renders the same  $CT$  in the model as the one derived in Section IV C, Eq. (33). Therefore, the initial Hamiltonian with the huge cutoff  $\Lambda$  is given in the SRG also by Eq. (32),

$$H_{\Lambda mn} = E_m \delta_{mn} - g_\Lambda \sqrt{E_m} \sqrt{E_n}. \quad (35)$$

The remaining discussion is focused on how one can evaluate  $H_\lambda$  using SRG equations in the model [1]. Next sections will describe the outcome of these calculations and what this outcome implies regarding the role of bound states in evaluation of effective theories with asymptotic freedom, or limit cycles. For completeness and reader's convenience, a perturbative SRG scheme that can be applied in evaluation of all Poincaré generators in QFT and used for the purpose of deriving dynamics of effective constituent quarks and gluons in QCD, is very briefly summarized in Appendix, Section VIII B.



### A. Generators of the similarity transformations

The SRG procedure is used here in its differential version. Below, prime denotes differentiation with respect to the parameter  $s = 1/\lambda^2$ , chosen for convenience.  $s$  increases from 0 to  $\infty$  when  $\lambda$  decreases from  $\infty$  to 0.

The effective Hamiltonians are generated by the formula

$$H'_\lambda = [T_\lambda, H_\lambda], \quad (36)$$

and  $T_\lambda$  is called the generator. The initial condition is set at  $\lambda = \infty$  (corresponding to  $s = 0$ ),

$$H_{\infty mn} = H_{\Lambda mn}, \quad (37)$$

using Eq. (35). Since all matrices in the model calculations, except for  $H_0$ , are functions of  $\lambda$ , the subscript  $\lambda$  from now on will be omitted everywhere, unless it is needed explicitly. The key point of the calculation is to choose the generator  $T$  in a way that can shed some light on the increase of an effective coupling constant  $g_\lambda$  at small  $\lambda$ .

One choice of  $T$  originates in the beautiful flow equation proposed by Wegner for Hamiltonian matrices in condensed matter physics [22, 23, 24]. A whole range of applications of Wegner's and similar equations in many-particle physics is reviewed in Ref. [25].  $T$  in Wegner's equation has the form

$$T_0 = [D, H], \quad (38)$$

where  $D$  denotes the diagonal part of the matrix  $H$ , or  $D_{mn} = D_m \delta_{mn}$  and  $V_{mn} = H_{mn} (1 - \delta_{mn})$ . An alternative equation is obtained with the choice

$$T_1 = [H_0, H], \quad (39)$$

which is employed in the SRG studies in nuclear physics [26, 27, 28]. The calculation concerning  $g_\lambda$  in asymptotically free theories (and limit cycle) that is reviewed here [1] is done with

$$T_f = [G, H], \quad (40)$$

where

$$G = fH_0 + (1 - f)D. \quad (41)$$

For  $f = 0$ , one has  $G = D$ , in which the diagonal part of interactions is fully included, and  $T = T_0$  of condensed matter physics. For  $f = 1$ , one has  $G = H_0$ , in which no interaction effects are included, and  $T = T_1$  of the nuclear studies. For intermediate values of  $f \in [0, 1]$ ,  $G$  includes interactions to an intermediate degree, correspondingly, and one can inspect what happens in various cases by varying  $f$ .

How does Eq. (36) work? Since  $T$  is a commutator of Hermitian matrices, it is anti-Hermitian and generates a unitary rotation of  $H$ , which means that traces of all powers of  $H$  are constant. In particular,

$$(Tr H^2)' = \left( \sum_{m=M}^N D_m^2 \right)' + \left( \sum_{m,n=M}^N |V_{mn}|^2 \right)' = 0. \quad (42)$$

This means that the off-diagonal matrix elements decrease if diagonal matrix elements increase, and vice versa. Eq. (36) implies

$$\left( \sum_{m=M}^N D_m^2 \right)' = 4 \sum_{mn} D_m (G_m - G_n) |V_{mn}|^2 \quad (43)$$

$$= 2 \sum_{mn} (D_m - D_n) (G_m - G_n) |V_{mn}|^2 \quad (44)$$

No negative terms appear on the right-hand-side when all differences  $D_m - D_n$  and  $G_m - G_n$  always satisfy the condition

$$(G_m - G_n)(D_m - D_n) \geq 0. \quad (45)$$

Using notation  $\Delta H_{0mn} = E_m - E_n$  and  $\Delta H_{Imn} = H_{Imm} - H_{Inn}$ , this condition can be rewritten, for every pair of diagonal elements number  $m$  and  $n$ , as

$$[f\Delta H_0 + (1-f)(\Delta H_0 + \Delta H_I)](\Delta H_0 + \Delta H_I) \geq 0. \quad (46)$$

Dividing by  $\Delta H_0 > 0$  for  $m > n$ , one obtains that if  $v = \Delta H_I/\Delta H_0$  satisfies the condition

$$[f + (1-f)(1+v)](1+v) \geq 0, \quad (47)$$

then the sum of squares of the diagonal matrix elements of  $H$  increases and the off-diagonal matrix elements of  $H$  decrease. (By the way, in the continuum limit for Hamiltonian matrices, such as  $b \rightarrow 1$ , Eq. (47) provides a condition on a derivative of the diagonal matrix elements of  $H_I$  with respect to diagonal matrix elements of  $H_0$ .)

For  $f = 0$ , the condition (47) says that  $(1+v)^2 \geq 0$ . This is always true and Wegner's generator always diagonalizes Hamiltonian matrices because the SRG evolution stops first when all  $|V_{mn}|^2$  are zero, except for elements  $V_{mn}$  for whose subscripts  $D_m = D_n$ ; these may in principle stay constant unless they change due to coupling with other non-vanishing off-diagonal matrix elements.

For  $f = 1$ , the condition (47) reads  $v \geq -1$ . This means that the diagonal part of the interaction must not decrease faster along the diagonal than the free energy increases. Convergence of  $H$  to a diagonal matrix may fail if

$$\frac{\Delta H_I}{\Delta H_0} < -1 \quad (48)$$

for some momenta. Since  $E_m$  increases monotonically with  $m$ , the lack of convergence may occur when  $H_{Imm}$  rapidly decreases with  $m$ . This happens when a negative matrix element appears on the diagonal among positive ones, leading to a negative eigenvalue that corresponds to a bound-state. The negative diagonal matrix elements on the diagonal do not guarantee that the SRG transformation stops driving off-diagonal matrix elements to zero, but it indicates that bound states may interfere with convergence of the SRG evolution of matrices  $H_\lambda$ .

For intermediate values of  $f$ , two sufficient, mutually exclusive but not necessary conditions for SRG evolution to bring  $H$  to the diagonal (outside regions of degeneracy mentioned earlier), are

$$\frac{\Delta H_I}{\Delta H_0} \leq \frac{1}{f-1} \quad \text{or} \quad \frac{\Delta H_I}{\Delta H_0} \geq -1. \quad (49)$$

In the model, it happens that these conditions can be violated when a bound state exists. The SRG evolution continues to bring  $H$  to the diagonal, but the effective coupling constant may become very large. This will be explained in the next section.

## VI. SOLUTIONS FOR $H_\lambda$

Formula (36) produces a set of coupled nonlinear differential equations for all matrix elements of  $H$ , which means  $(N - M + 1)^2$  functions of  $s$ ,

$$D'_n = 2 \sum_{k=M}^N (G_n - G_k) V_{nk} V_{kn}, \quad (50)$$

$$\begin{aligned} V'_{mn} = & -(G_m - G_n)(D_m - D_n)V_{mn} \\ & + \sum_{k=M}^N (G_m + G_n - 2G_k)V_{mk}V_{kn}. \end{aligned} \quad (51)$$

This is a formidable set of equations to solve. For example, if  $b = 2$ , one needs 37 states to span the energy range between  $b^M \sim 1$  keV and  $b^N \sim 70$  TeV, and this implies 703 functions of  $s$  for real symmetric matrices (a complex Hermitian Hamiltonian would imply 1369 functions). The only known way to learn precisely how the solutions to these equations look like with the initial condition at  $s = 0$  set by Eq. (37), is to solve them numerically.

Fortunately, the numerical analysis (it is too extensive to review here; readers interested in the numerical analysis need to consult the original literature) produces results that can be summarized by a very simple, qualitatively accurate analytic formula for  $H_\lambda$

$$H_{\lambda mn} \sim \left[ E_m \delta_{mn} - g_\lambda \sqrt{E_m} \sqrt{E_n} \right] e^{-(E_m - E_n)^2 / \lambda^2}, \quad (52)$$

which becomes exact in the limit  $(E_m + E_n)/\lambda \rightarrow 0$ . This means that the matrix elements evolve with  $\lambda$  in a coherent fashion and a relatively small number of simple functions of  $E_m$  and  $E_n$  is sufficient to reasonably well describe the evolution of all of them. In particular, for  $E_m$  and  $E_n \ll \lambda$ , the exponential factor is equivalent to 1 and SRG evolution of the entire low-energy corner of the Hamiltonian matrix is described by just one function denoted by  $g_\lambda$  in Eq. (52). In analogy with the Thomson limit in QED, this function can be extracted from the lowest energy diagonal matrix element of the interaction term in  $H_\lambda$ ,  $H_{\lambda MM} = E_M - g_\lambda E_M$ . The result is that the following quantity is called the effective coupling constant in a Hamiltonian with a finite SRG cutoff  $\lambda$ ,

$$g_\lambda = 1 - H_{\lambda MM}/E_M. \quad (53)$$

An alternative definition of  $g_\lambda$ , with the same result, could be based on the interaction matrix elements between two basis states corresponding to the eigenvalues  $E_M$  and  $E_{M+1}$ , in analogy with the example illustrated by Eq. (16). Namely,

$$g_\lambda = -H_{\lambda MM+1}/\sqrt{E_M E_{M+1}}. \quad (54)$$

### A. Asymptotically free effective interactions for different generators

The effective coupling constant defined in Eqs. (53) or (54), is a function of  $\lambda$ . The observation made in Ref. [1] is that the function one obtains depends on the choice of the parameter  $f$  in the generator of the SRG transformations. A generic example is shown in Fig. 1. In this example,  $b = 4$ ,  $N = 16$ ,  $M = -25$ , and  $g_\infty = g_\Lambda \sim 4/100$ , where  $\Lambda = 4^{16} \sim 4 \cdot 10^9$ . The bound-state energy,  $E_B \simeq 8 \cdot 10^{-6}$ .

The mathematical mechanism of SRG transformations by which the effective coupling constant  $g_\lambda$  increases to huge values for  $f = 1$ , or stays near 1 for  $f = 0$ , or increases to intermediate values for intermediate values of  $f$ , is explained in detail in Ref. [1]. The physical essence of the explanation is that SRG transformations with different values of  $f$  place eigenvalues of  $H_\lambda$  on the diagonal when  $\lambda \rightarrow 0$  in different places. A natural location on the diagonal for an eigenvalue  $E$  would be a diagonal matrix element in the basis state  $|m\rangle$  to which  $H_0$  assigns the eigenvalue  $E_m \sim E$ .

When  $f = 0$ , interactions are fully accounted for in the generator through  $G$ , see Eq. (41). The reduction of  $\lambda$  brings  $H_\lambda$  nearer its diagonal including the interaction energy that is responsible for binding. So, some diagonal matrix element can naturally become negative for some basis state  $|m\rangle$  with  $E_m \sim |E| = E_B$ .

When  $f = 1$ , interactions are completely ignored in  $G$ , i.e.,  $G = H_0$ . The diagonal matrix elements can still approach 0 from above. But at some point, the monotonic increase of diagonal matrix elements along the diagonal

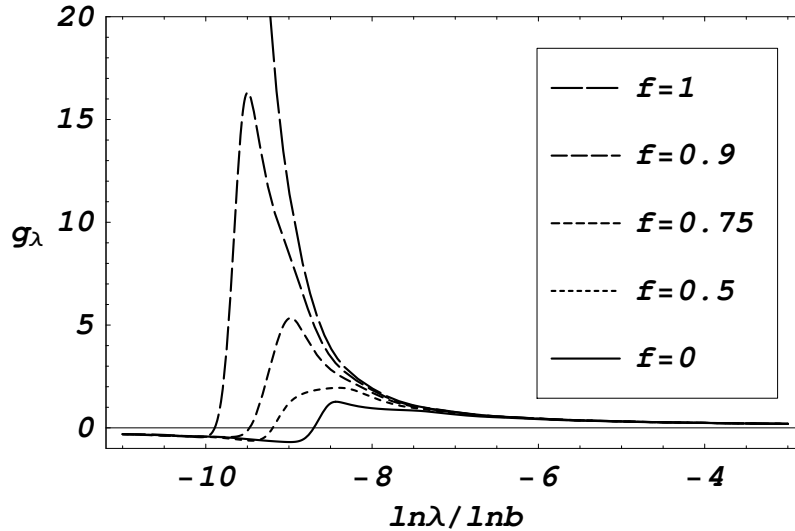


FIG. 1: Rise of the asymptotically free coupling constant  $g_\lambda$  at small  $\lambda$ , drawn as a function of  $\ln \lambda / \ln b$  for 6 values of  $f$ :  $f = 0$ , 0.5, 0.75, 0.9, and 1 (the larger  $f$ , the higher curve). The rapid increase of  $g_\lambda$  for  $f = 1$  below  $\ln \lambda / \ln b \sim -8$ , occurs because  $\lambda$  decreases below the scale of binding energy  $E_B$ . When  $\lambda \rightarrow 0$ ,  $g_\lambda$  for  $f = 1$  does not actually reach  $\infty$  but  $|E_B|/b^M \sim 10^{10}$ . For  $f = 0$ , the huge increase of  $g_\lambda$  is absent and instead  $g_\lambda$  never exceeds order 1. See the text for explanation.

toward large energies (the greater subscript  $m$  of a rotated basis state the greater the diagonal matrix element) is broken and a decrease along the diagonal (instead of increase) occurs at some state  $|m\rangle$ . At this place, the SRG transformation with  $f = 1$  stops reducing the size of the nearby off diagonal matrix elements. Instead, their size increases and maintains the strength required for reproduction of the bound-state eigenvalue on the diagonal for some state  $|m\rangle$  with  $E_m < E_B$ . The shift of the eigenvalue  $-E_B$  of a fixed magnitude to a lower-energy basis state requires a corresponding increase of the interaction strength. For  $f = 1$ , the shifting of  $E = -E_B$  toward low energies continues to the very end of the SRG evolution at  $\lambda = 0$  and places  $E = -E_B$  at the state with the lowest allowed  $E_m$  in the model: the bound-state eigenvalue appears as the diagonal matrix element  $\langle M|H_\lambda|M\rangle$  when  $\lambda \ll b^M$ . As a result of this cumulative shift away from the natural momentum scale for the bound-state wave function, the coupling constant  $g_\lambda$  continues to increase in order to eventually produce  $-E_B$  on the diagonal through  $(1 - g_\lambda)b^M$ . Thus,  $g_\lambda$  increases toward  $E_B/b^M$ , which is a huge number for a large negative  $M$ .

When  $f$  has an intermediate value between 0 and 1, the cumulative shift stops at certain state  $|m\rangle$ , for which  $fE_m + (1 - f)(1 - g_\lambda)E_m$  becomes negative and reproduces the eigenvalue  $-E_B$ . This requires  $g_\lambda$  with  $\lambda \sim E_m$  to increase only to  $\sim E_B/E_m$ .

The above scenario of how the increase of  $g_\lambda$  occurs, depending on the value of  $f$ , is reflected in Fig. 1. Maximum of the curve with  $f = 0.75$  occurs at  $-9$ , instead of  $-8.5$  in the case of  $f = 0$ , and the maximal value of  $g_\lambda$  for  $f = 0.75$  is 4, or  $b$  times larger than in the case with  $f = 0$ . The maximum of the curve with  $f = 0.9$  occurs at  $-10$ , instead of  $-8.5$  in the case of  $f = 0$ , and the maximal  $g_\lambda$  for  $f = 0.9$  is 16, or  $b^2$  times larger than in the case with  $f = 0$ , and so on. The case with  $f = 1$  leads to apparently indefinite and accelerating increase that smoothly continues the inverse logarithmic growth that characterizes asymptotic freedom above the scale of binding. The transition from an asymptotically free RG behavior to the behavior dominated by bound states occurs when  $g_\lambda$  crosses 1, cf. Section IV D.

In summary, the model demonstrates a possibility that an unlimited increase in the asymptotically free interaction at small energies is caused by omission of interactions responsible for the existence of a bound state in the generator of the SRG transformations. When the generator fully accounts for the interactions responsible for binding, the magnitude of the coupling constant never significantly exceeds 1.

The model example is potentially important because it suggests that the increase of the coupling constant in asymptotically free theories that is observed in perturbation theory (corresponding to  $G = H_0$  and  $f = 1$ ) may be not related to the phenomenon of confinement but to the phenomenon of formation of bound states. The latter is in principle a simpler one to handle in theory than the former. If it were indeed the case, application of the SRG procedure to QCD may help theorists in penetrating the range of momentum scales near  $\Lambda_{QCD}$  and explaining hadronic states without immediate necessity to solve the problem of confinement that is relevant at distances much larger than the size of individual hadrons (see also Appendix, Section VIII B).

## B. Comment on asymptotic freedom and limit cycle

An apparently very small alteration of the model introduced in Section II leads to new ways of thinking about asymptotic freedom in terms of limit cycle. RG limit cycles were discovered in the context of strong interactions in Ref. [16], using the nowadays standard RG procedure. The SRG approach to limit cycle is based on Ref. [2]. The initial Hamiltonian matrix of the altered model is

$$H_{\Lambda mn} = \sqrt{E_m E_n} [\delta_{mn} - g_\Lambda - ih \operatorname{sgn}(m - n)] . \quad (55)$$

where  $i = \sqrt{-1}$  and the new coupling constant  $h$  can be an arbitrarily small real number. If  $h$  is an arbitrary irrational number, the model typically exhibits chaotic RG behavior. When  $h = \tan \frac{\pi}{p}$  with  $p$  an integer greater than 2, a limit cycle occurs. Namely, repeating the same procedure as described in the previous sections for  $h = 0$ , one finds that the coupling constant  $h$  does not evolve with  $\lambda$  at all, while the coupling constant  $g_\lambda$  that replaces  $g_\Lambda$  in a similar way as  $g_\lambda$  replaced  $g_\Lambda$  for  $h = 0$ , oscillates with  $\lambda$  with a multiplicative period  $b^p$ . This means that  $g_{\lambda_1}$  has the same value as  $g_{\lambda_2}$  if  $\lambda_1 = (b^p)^k \lambda_2$  with an arbitrary integer  $k$ . The periodicity (cycle) of the coupling constant is associated with existence of infinitely many bound states (in the limit  $M \rightarrow -\infty$ ) whose binding energies form a geometric series converging on zero with quotient  $1/b^p$ .

RG evolution of an asymptotically free interaction forms a part of a limit cycle for all values of  $\lambda$  for which  $h$  is very small in comparison to  $g_\lambda$ . In this range,  $h$  does not matter and the Hamiltonians of the model defined by Eq. (55) evolve in the same way as in the case of Eq. (35) (see Fig. 3 in Ref. [2]). Consider the bottom-up evolution in which  $\lambda$  grows and the coupling constant  $g_\lambda$  decreases as an inverse of a logarithm of  $\lambda$  until it becomes comparable with  $h$ . Then, instead of  $g_\lambda$  continuing its logarithmic falloff indefinitely,  $h$  takes over and forces  $g_\lambda$  to change sign and subsequently grow in size. This continues until a new high-energy bound state is reconstructed. When this happens,  $g_\lambda$  switches sign to positive again and grows rapidly above 1, its magnitude depending on the size of  $f$ . The switch

occurs at the scale  $\lambda$  on the order of the energy (momentum) where the new bound-state energy is located on the diagonal. Further increase of  $\lambda$  produces a falloff like in asymptotic freedom again, until  $g_\lambda$  decreases again down to the size of  $h$ .

The range of scales for which the cycle looks like asymptotic freedom is given by the period of the cycle, characterized by the factor  $e^{\pi/h}$ , which can be very large when  $h$  is very small. Such behavior of the simple model is of general interest because it suggests that the hierarchy problem may stem from continuing perturbative analysis for coupling constants  $g_\lambda \ll 1$  while overlooking formation of new generation of bound states due to very small, and so far unknown, interactions of the type represented by the coupling constant  $h$ . But in order to see their presence, one has to use SRG procedure with the generator that includes interactions in  $G$ .

## VII. CONCLUSION

The simple model study shows that the SRG procedure may be a suitable tool to handle the increase of the coupling constant  $g_\lambda$  in QCD when  $\lambda \rightarrow \Lambda_{QCD}$ . If the generator of SRG transformations does not include interactions in  $G$ ,  $f = 1$  in Eq. (41), the effective coupling constant increases to very large values quickly as soon as the SRG scale parameter  $\lambda$  becomes comparable with the momentum scale that characterizes formation of a bound state. In QCD, the corresponding momentum scale would be much larger than the scale associated with confinement because the size of a single hadron is much smaller than the distances at which confinement matters. If the generator of SRG transformations includes interactions in  $G$ , the SRG parameter  $\lambda$  can be brought down to the momentum scale that characterizes bound states and the coupling constant does not increase to large values. These model results suggest that the SRG procedure should be applied to QCD because it may offer help in understanding the binding mechanism for quarks and gluons using expansion of  $H_{\lambda QCD}$  in powers of an effective coupling constant and without need for prior understanding of confinement. Interestingly enough, the SRG procedure may also help us establish a connection between asymptotic freedom and limit cycle. In this respect, the model shows that in order to handle the case of limit cycle the generator of SRG transformations must include interactions in  $G$ .

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## VIII. APPENDIX

The Appendix describes examples of QFT counterparts of the concepts concerning RG procedure and bound states that are introduced in the main text in a simple model.

### A. Potentials with $\delta$ -functions

When one calculates corrections to the Coulomb potential in QED, one obtains a  $\delta$ -function as the Uehling term in  $H_I$ . This term contributes a small part in the Lamb shift. The Coulomb potential,  $-\alpha/r$ , in the Schrödinger equation is changed to [29]

$$V(\vec{r}) = -\frac{\alpha}{r} - \frac{4\alpha^2}{14m_e^2} \delta^{(3)}(\vec{r}) \quad (56)$$

The correction appears suppressed by  $\alpha \sim 1/137$  in comparison to the Coulomb potential. In first-order perturbation theory, one obtains truly tiny corrections ( $\sim 10^{-7}$  eV for  $2S$  states [29]). However, the second order correction involves multiplication of the  $\delta$ -function by itself and produces infinity. The problem is not merely due to the use of perturbation theory instead of solving the Schrödinger equation exactly since the  $\delta$ -function (or a similarly singular function) leads to so strong a potential that the wave function collapses unless the singular function is somehow replaced by a regular one. One can attempt to derive a regular expression in perturbation theory, say, by limiting the range of momenta in intermediate states from above by the electron mass  $m_e$  times the speed of light. On the other hand, a complete analysis should include the formation of a bound state, and bound states are not describable

in perturbation theory. Some form of an effective theory is necessary [30] (see also Ref. [31]). In contrast with other approaches, the singular potentials that are obtained in QCD (or QED) in the SRG approach, are always effectively regulated by form factors of width  $\lambda$ , the width playing the role of a renormalization group parameter (e.g., see Ref. [32]).

Studies of  $\delta$ -function potentials by physicists have a long history [33, 34, 35, 36, 37, 38, 39, 40, 41] and there exist mathematical textbooks on the subject [42, 43]. The simple model used in this lecture can also be derived by discretizing momentum on a logarithmic scale in the  $S$ -wave Shrödinger equation for a particle moving on a plane in the presence of a potential proportional to a  $\delta$ -function (see the original literature).

## B. SRG procedure in QFT

Since the model discussed in this lecture appears deceptively simple, this section provides a telegraphic overview of how the SRG procedure applies in QFT. The original SRG application to QCD was outlined in Ref. [44], for calculating matrix elements of light-front (LF) effective  $H_{\lambda QCD}$  using perturbation theory. A perturbative calculus for  $H_{\lambda QCD}$  with  $\lambda \gg \Lambda_{QCD}$  in terms of creation and annihilation operators, was developed in Ref. [45], and shown in Ref. [46] to be able to produce in simple scalar theories not only the Hamiltonian operator (the energy-momentum tensor component  $T_{\lambda}^{+-}$ ) but also other generators of the Poincaré algebra. This calculus produced LF QCD Hamiltonians with running coupling constant  $g_{\lambda}$  and recently led to a reasonable description of heavy-quarkonium spectra, still using crude and as yet unverified approximations concerning terms order  $g_{\lambda}^4$  [32]. The calculus is invariant with respect to 7 independent Poincaré symmetries (including 3 boosts), satisfies required cluster property [31], and guarantees that the resulting Hamiltonians  $H_{\lambda}$  have form factors in the interaction vertices of width  $\lambda$  in energy, the width being the SRG parameter. All these features are required when one attempts to derive the parton model and spectroscopy of hadrons from one and the same Hamiltonian formulation of QCD.

The point of departure is a canonical LF Hamiltonian in QFT, which requires regularization and counterterms,

$$H = [H_{can} + H_{CT}]_{reg} . \quad (57)$$

The canonical gluon field  $A$  in  $A^+ = 0$  gauge (LF in the Minkowski space-time is defined by the condition  $x^+ = x^0 + x^3 = 0$  and  $A^+ = 0$  means that  $A^0 + A^3 = 0$ ) and quark field  $\psi$ , are expanded into their Fourier components at  $x^+ = 0$ . The Fourier components correspond to canonical creation and annihilations operators for bare quarks and gluons (or other bare particles in other theories than QCD). These canonical operators, say  $q_{can}$ , are related by a unitary transformation  $U_{\lambda}$  to their counterparts for effective particles, say  $q_{\lambda}$ ,

$$q_{\lambda} = U_{\lambda} q_{can} U_{\lambda}^{\dagger}, \quad (58)$$

and  $U_{\infty} = 1$ . The Hamiltonian  $H = H(q_{can})$  is assumed equal to  $H_{\lambda}(q_{\lambda})$ , but the coefficients in expansion of  $H_{\lambda}(q_{\lambda})$  in powers of  $q_{\lambda}$  are different from the coefficients of expansion of  $H(q_{can})$  in powers of  $q_{can}$ . The object of the SRG calculation are the coefficients. One actually works in the constant basis in the operator space and evaluates  $\mathcal{H}_{\lambda} = H_{\lambda}(q_{can})$ . The SRG equation is

$$\frac{d}{d\lambda} \mathcal{H}_{\lambda} = [\mathcal{T}_{\lambda}, \mathcal{H}_{\lambda}], \quad (59)$$

where

$$\mathcal{T}_{\lambda} = \mathcal{U}'_{\lambda} \mathcal{U}_{\lambda}^{\dagger}, \quad (60)$$

and  $\mathcal{U}_{\lambda}$  corresponds to  $U_{\lambda}$ . Assuming the initial condition of Eq. (57),

$$\mathcal{H}_{\infty} = [H_{can} + H_{CT}]_{reg}, \quad (61)$$

one derives  $H_{\lambda}$  from the formula

$$\mathcal{H}_{\lambda} = \mathcal{H}_{\infty} + \int_{\infty}^{\lambda} d\omega [\mathcal{T}_{\omega}, \mathcal{H}_{\omega}] \quad (62)$$

order by order in perturbation theory, eventually replacing  $q_{can}$  by  $q_{\lambda}$ .

The key is  $\mathcal{T}_{\lambda}$ .  $\mathcal{H}_{\lambda}$  is split into a bilinear term in  $q_{can}$ , denoted by  $\mathcal{H}_1$ , and the remaining terms, denoted by  $\mathcal{H}_2$ .  $\mathcal{H}_2$  is assumed to contain the form factors  $f_{\lambda}$  in vertices (the letter  $f$  used here has nothing to do with the letter  $f$

used in  $G$  in Section V A). Thus,  $\mathcal{H}_2 = f\mathcal{G}_2$ , and the form factor is defined for all operators in the same way as in the following example:

$$\mathcal{G} = \int [123] g(1, 2, 3) q_{can1}^\dagger q_{can2}^\dagger q_{can3}, \quad (63)$$

$$f\mathcal{G} = \int [123] f(123) g(1, 2, 3) q_{can1}^\dagger q_{can2}^\dagger a_{can3}, \quad (64)$$

$$f(123) = \exp [-(\mathcal{M}_{12}^2 - \mathcal{M}_3^2)/\lambda^4]. \quad (65)$$

Namely, when an operator  $\mathcal{G}$  contains a product of  $m$  creation and  $n$  annihilation operators, the form factor operation inserts the same function  $f$  of the difference between invariant masses squared,  $\mathcal{M}_m^2 = (\sum_{i=1}^m p_i)^2$  and  $\mathcal{M}_n^2 = (\sum_{j=1}^n k_j)^2$ , that are associated with the particle momenta using eigenvalues of  $\mathcal{G}_1$ . The SRG generator in QFT is defined by the commutator [45]

$$[\mathcal{T}, \mathcal{G}_1] = [(1 - f)\mathcal{G}_2]'. \quad (66)$$

This choice guarantees that the resulting interactions are connected (cluster property) and perturbation theory for evaluation of  $\mathcal{H}_\lambda$  never contains small energy denominators, provided the function  $1 - f$  vanishes faster than linearly in the difference of invariant masses (for the function adopted in Eq. (65),  $1 - f$  vanishes quadratically).

### C. Questions concerning SRG and AdS/CFT in QCD

A few questions are posed here regarding the possibility that the SRG procedure in QCD and a RG interpretation of AdS/CFT duality in the context of QCD, can be related. These questions reflect how little is understood about the SRG procedure and formation of bound states in QCD.

SRG procedure is expected to render Hamiltonians  $H_\lambda$  whose eigenstates in the case of LF QCD (as a part of the standard model) can be used to define hadrons and calculate hadronic observables. A hadron eigenstate is a linear combination of Fock components  $|n\rangle$  with various values of  $n$ , where  $n$  denotes the number of operators  $q_\lambda^\dagger$  in a product that is used to create the state  $|n\rangle$  from the LF vacuum state  $|0\rangle$ . The corresponding wave functions  $\psi_{n\lambda}$  will depend on  $\lambda$  while the eigenvalue and the eigenstate as a whole will not. Expansion of a hadron state into components with various numbers of effective constituents will be very broad in energy for  $\lambda \gg \Lambda_{QCD}$  (corresponding to a parton model state for probes acting with large energy transfers) and may be dominated by the lowest constituent-number components for  $\lambda \sim \Lambda_{QCD}$  (corresponding to the constituent model for probes acting with small energy transfers).

The family of Hamiltonians  $H_\lambda$ , all equivalent, can be imagined to be related to a family of complex effective classical Lagrangian densities  $\mathcal{L}(\lambda, x)$  in the Minkowski space-time, all of them representing the same theory. The parameter  $\lambda$  can be considered a 5th dimension in a larger theory and one can contemplate actions of the form

$$S = \int d\lambda d^4x \mathcal{L}(\lambda, x). \quad (67)$$

In such action, the coupling constant  $g_\lambda$  can be considered a function of the 5-th coordinate  $\lambda$ , and also a function of  $x$ . Eventually, no hadronic observable will depend on  $\lambda$ . Nevertheless, a rich dynamics could be considered in the 5-dimensional space, or in more dimensions, in which the family of equivalent Lagrangians labeled by  $\lambda$  could emerge as describing configurations that dominate appropriate path integrals.

Suppose the dependence on  $\lambda$  is analogous to Polyakov's RG interpretation of the 5th (Liouville) dimension in string theory [5, 6], cf. [4]. In the SRG procedure, however, one always deals with effective quantum field theories in the Minkowski space. Can these effective theories explicitly include scale dependent gluonic string dynamics that could be “a shadow on the wall” of what happens in the bulk? According to [32], one may expect explicit formation of quantum gluonic strings in LF Hamiltonian QCD in the Minkowski space when one considers eigenstates in which relative motion of constituents corresponds to considerably larger energies than  $\Lambda_{QCD}$  and components with  $n \gg 1$  for effective gluons are important. Could they correspond in some way to a string theory in the bulk?

There is another line of thinking that leads to similar questions, based on the observation that the LF wave functions  $\psi_{n\lambda}$  can be used to calculate hadronic form factors using formulae similar to the ones considered by Brodsky and de Teramond in Refs. [9, 10], with additional factors that will result from using the SRG procedure. Can one connect the holography proposed by Brodsky and de Teramond with the  $\lambda$ -dependent hadronic constituent distributions (densities obtained using sums of moduli squared of the LF wave functions  $\psi_{n\lambda}$ ), additional SRG factors, and off-energy-shell

old-fashioned scattering amplitudes for effective constituents with form factors of width  $\lambda$ , all of which would result from  $H_{\lambda QCD}$  obtained from the SRG procedure?

Such questions will not be easily answered. For example, there is a potential complication involved in QCD due to an infrared limit cycle [47]. On the other hand, the model described here shows that the SRG procedure can deal also with limit cycles by including interactions in the generator.

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